

# Fast mode of rotating atoms in one-dimensional lattice rings

T. Wang<sup>1</sup> and S. F. Yelin<sup>1,2</sup>

<sup>1</sup>*Department of Physics, University of Connecticut, Storrs, CT 06269*

<sup>2</sup>*ITAMP, Harvard-Smithsonian Center for Astrophysics, Cambridge, MA 02138*

(Dated: February 1, 2008)

We study the rotation of atoms in one-dimensional lattice rings. In particular, the “fast mode”, where the ground state atoms rotate faster than the stirring rotating the atoms, is studied both analytically and numerically. The conditions for the transition to the fast mode are found to be very different from that in continuum rings. We argue that these transition frequencies remain unchanged for bosonic condensates described in a mean field. We show that Fermionic interaction and filling factor have a significant effect on the transition to the fast mode, and Pauli principle may suppress it altogether.

Recent progress in manipulating neutral atoms includes paired states of fermions, the crossover between a Bose-Einstein condensate (BEC) molecules and a Bardeen-Cooper-Schrieffer (BCS) superfluid of loosely bound pairs [1, 2, 3, 4, 5, 6, 7]. So far, most of the experimental studies involving cold atoms were conducted in continuum. Optical lattices, however, allow many effects of interest: Superfluid to Mott insulator transition [8], Bloch oscillation of particles in lattices due to Bragg scattering [9], parametric atomic down conversion in BEC [10], etc.

By rotating atoms in a continuum, many phenomena have already been investigated: the appearance of vortices [11, 12] in both the BEC and BCS [6], quantum Hall states for fermions with fast rotation frequency [13] and vortex lattices in the lowest Landau level for BECs [12, 14]. Experimentally, these rotations are realized by stirring the cold atoms. When the atoms are rotated in lattices, theoretical studies show that lattices lead to many new effects under rotation, such as structural phase transitions of vortex matter [15]. Also, near the superfluid–Mott insulator transition, the vortex core has a tendency toward the Mott insulating phase [16] and second-order quantum phase transitions between states of different symmetries in a two dimensional (2D) lattice were observed at discrete rotation frequencies [17, 18]. In particular, we have recently shown [19] that it is possible for atoms to stay in a 2D lattice even if the rotation frequency is larger than the harmonic trapping frequency and density depletion in the trap can then be developed in such regime.

However, in a 2D lattice, analytic results are not easy to obtain. The origin of some phenomena is thus hard to catch. The so-called fast mode for rotating atoms is such an example. This mode was shown to exist in 2D lattices [17, 18], but we investigate this here in a 1D lattice through both analysis and numerical simulations. Throughout this article, we define a current to be “positive” if it is faster than the stirring force and “negative” if it is slower. Thus, the ground state (GS) is in the “fast mode” if its current is positive. While the ring lattice is much simpler, it catches much of the physics of the fast mode in higher dimensions, too. Specifically, we

will study the origin of the fast mode, its dependence on quantum statistics and whether the fast mode is always present when the rotation frequency is large enough. Before we move on, we remark that stirring a classical fluid can never achieve positive current in the rotating frame and thus the fast mode does not exist for a classical fluid.

For completeness, we first review the rotation of a single quantum particle in a continuum ring with radius  $R$ . In the rotating frame, the Hamiltonian for such system is  $H = -\frac{\hbar^2}{2mR^2}\frac{\partial^2}{\partial\phi^2} - \Omega L_z$ , where  $L_z$  is the angular momentum operator  $-i\hbar\partial/\partial\phi$ ,  $m$  is the mass,  $\phi$  is the angular coordinate and  $R$  is the radius of the ring.  $H$  can be reduced to  $H = -\frac{\hbar^2}{2mR^2}(\frac{\partial}{\partial\phi} - i\frac{m\Omega R^2}{\hbar})^2 - m\Omega^2 R^2/2$  with eigenfunctions  $\psi_n(\phi) = \exp(in\phi)/\sqrt{2\pi}$ , where  $n$  are integers. Except  $m\Omega^2 R^2/2$  in  $H$  which obviously reflects the centrifugal potential, the Hamiltonian is identical to that of a  $q$ -charged particle put into a ring thread by a flux  $\phi = 2\pi m\Omega R^2 c/q$  [20]. The spectrum  $E_n = \frac{\hbar^2}{2mR^2}(n - \frac{m\Omega R^2}{\hbar})^2 - m\Omega^2 R^2/2$  of the Hamiltonian shows that when  $m\Omega R^2/\hbar$  is a half integer  $k+1/2$ , all energy levels become two-fold degenerate,  $E_{k+l+1} = E_{k-l}$ ; For all other values of  $\Omega$ , the ground states are not degenerate. Furthermore, when  $m\Omega R^2/\hbar$  is a half integer, the doubly degenerate ground states ( $n = 0, 1$ ) can have either positive particle currents (as is the case for  $n = 1$ ) or negative particle current ( $n = 0$ ) in the rotating frame. For all other cases, the non-degenerate ground states ( $n = 0$ ) have negative currents in the rotating frame. In what follows, we will show that lattices change many of these results.

We now study the rotation of bosonic and fermionic atoms in a lattice ring using a Hubbard model. In the rotating frame, the single band Boson Hubbard Hamiltonian is [17, 18, 21]

$$H = (\sum_{\langle i,j \rangle} (-t - i\Omega K_{i,j}) b_i^\dagger b_j + H.c.) + U_B \quad (1)$$

where  $b_i^\dagger$  ( $b_i$ ) is the Boson creation (annihilation) operator at site  $i$ , and the modified single band Fermion Hubbard Hamiltonian is [17, 18, 21]

$$H = \left( \sum_{\langle i,j \rangle, \sigma} (-t - i\Omega K_{i,j}) c_{i,\sigma}^\dagger c_{j,\sigma} + H.c. \right) + U_F \quad (2)$$

where  $c_{i,\sigma}^\dagger$  ( $c_{i,\sigma}$ ) is the creation (annihilation) operator for a fermion of spin  $\sigma = \uparrow, \downarrow$  at site  $i$ . In the Eqs. (1-2),  $\langle i, j \rangle$  indicates nearest neighbor pairs,  $t$  is the hopping term between neighboring sites, and  $H.c.$  means Hermitian conjugate. The interaction term for bosons (fermions) is  $U_B = U \sum_i (b_i^\dagger b_i - 1) b_i^\dagger b_i$  ( $U_F = U \sum_i c_{i,\uparrow}^\dagger c_{i,\uparrow} c_{i,\downarrow}^\dagger c_{i,\downarrow}$ ) with  $U$  the interaction strength determined by the s-wave scattering length and the lattice potential. With equally spaced sites as exemplified by this paper, the geometric factor  $K_{i,j}$  is given by  $K = \beta \sin \alpha/2$ , where  $\alpha$  is the angle subtended by neighboring sites with respect to the axis of rotation, and  $\beta$  is a dimensionless constant of order 1 characterizing the lattice geometry and depth [17, 18, 19]. In Eqs. (1,2), the lattice constant and  $\hbar$  are set to be one, so energies are given in units of the hopping energy. Note that the Hamiltonian used in Eqs. (1,2) is an approximation based on a perturbative treatment of  $\Omega L_z$ : The Wannier basis states are the eigenstates of the Hamiltonian without this term. While  $\Omega L_z$  may be included for an alternative (and possibly better) Wannier basis, Ref. [18] (e.g., Fig. 3 therein) shows that the differences are negligible if  $\Omega$  is small. This, however, is the regime we are interested in here.

As a building block for the following study, a single atom in the lattice with  $N_A$  sites is studied first. Since wavefunctions have to be unique, the total winding phase  $\Theta$  around the ring has to be a multiple of  $2\pi$ , e.g.,  $\Theta(n) = n2\pi$  with  $n$  a nonnegative integer determining the symmetry of the state. The wavefunctions in an evenly distributed lattice site system are totally determined by the phase  $\phi(n) = \Theta(n)/N_A$  as  $|\psi\rangle = \sum_j \exp(i\phi_j)/\sqrt{N_A} |j\rangle$ , where  $|j\rangle$  is the state when the particle is at site  $j$ . We can then use the wavefunctions to determine their energies, currents (see Fig. 1) etc.. In particular, the energies for the low lying states are  $E(n) = -2(t \cos \phi(n) + K\Omega \sin \phi(n))$ , a linear function of rotation frequency  $\Omega$  for any fixed  $n$ . From  $E(n)$ , we see that for  $K\Omega \gg t$ , the ground state has  $n = n_m \equiv \lfloor N_A/4 \rfloor$ , where  $\lfloor x \rfloor$  is the largest integer number that is less than or equal to the specified number  $x$ . This means that any state with maximum phase differences of  $\pi/2$  between neighboring sites are ground state for large  $\Omega$ . Furthermore, the linear dependence of  $E(n)$  on  $\Omega$  should be contrasted to the quadratic dependence in continuum rings. Since the discrete rotational symmetry is not broken by the rotation (the  $\Omega$  term in Eqs. (1-2)), states with different symmetry experience level crossings as  $\Omega$  changes. By comparing the energies of states with different  $n$ , the level crossings between these states can be determined analytically. Figure 1a is an example of the energies and the level crossings for the low lying states of an atom in an 8-site ring lattice. These level crossings demonstrate how the system evolves between states of different rotational symmetries as  $\Omega$  increases [17, 18].

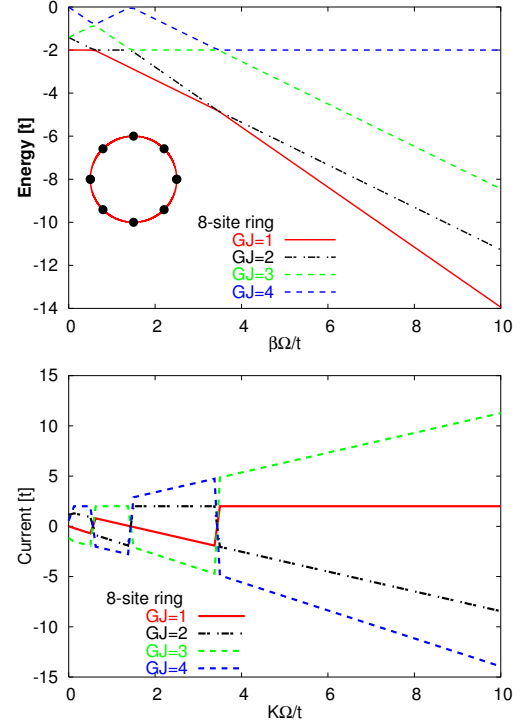


FIG. 1: Analytically calculated energies (a) currents (b) of the low lying states of an atom in the ring lattice. The inset in (a) shows a sketch of a rotating ring lattice with eight evenly distributed sites. “GJ” means the index of the low lying states. For example, GJ=1 is for the ground state and GJ=2 is for the first excited state, and so on. At  $\Omega > \Omega_c$ , the GS current is a constant  $2t$ , because the phase difference  $\pi/2$  between neighboring-sites makes  $\Omega$  dependent term in Eq. (3) have no contribution to the current.

Corresponding to these level crossings, the current and other observables change as well. As an important observable, the current is used in what follows to study the fast mode in the lattice ring. The currents are shown in Figure 1b, where the particle current between site  $i$  and  $j$  is calculated by [17, 18]

$$J_{ij} = i[n_i, H_{ij}] = it(a_i a_j^\dagger - H.c.) + K\Omega(a_i a_j^\dagger + H.c.) \quad (3)$$

for each individual state. All the currents  $J$  plotted in this paper are the integrated currents along the ring. In a homogeneous ring, they are given by  $J = N_s J_{ij}$ . It is not surprising to see the persistent currents of the excited states can be bigger than that of GS or have opposite direction. The excited states will be used later to construct the state of spin polarized fermions.

The fast mode in the lattice ring happens when the winding number of GS is at its maximum  $n_m$ , where neighboring sites have maximum phase difference  $n_m 2\pi/N_A$  and thus maximum currents are achieved. By matching the energy of the state with winding number  $(n_m - 1)$  with the energy of the state with winding number  $n_m$ , we obtain  $\Omega_c K(1 - \cos 2\pi/N_A) = t \sin 2\pi/N_A$  to determine the rotational frequency  $\Omega_c$  above which GS is

in the fast mode. Setting  $N_A = 4$  reproduces the special result in Ref. [17, 18]. From our general result, it can be seen that for larger  $N_A$ ,  $\Omega_c$  is higher to achieve the fast mode, which makes it difficult to be realized experimentally. In addition, a constant current of  $2t$  is achieved when the phase difference of  $\pi/2$  between neighboring sites makes the current independent of  $\Omega$  (see Eq. (3) and Fig. 1). For comparison, particles in continuum rings can move faster than the stirring only when  $m\Omega R^2/\hbar$  is a half integer. The difference is because the lattice breaks the continuous rotational symmetry.

A single atom in a few-site ring does not answer the question of whether the fast mode comes from finite number of atoms or finite number of sites. Thus we study the same ring but with more than one atom, in which case the quantum statistics of the atoms comes into play. First we consider a BEC consisting of many zero-temperature bosons described by a mean field. When the ring is homogeneous, the distribution of the atoms in the lattice does not depend on the rotation. This means that the mean field wavefunction is the same as that of the single-atom wavefunction except normalized to  $N$  atoms, and thus the transition to fast mode does not change. Since for a BEC many bosons are involved, the fast mode is therefore only due to finite number of sites in the lattice ring.

Spin polarized fermions at zero temperature show more interesting dynamics. The GS current of  $N$  spin polarized fermions can be obtained by summing over the currents of the lowest  $N$  states of a single particle in the lattice, because these fermions, assuming no p-wave interaction, occupy the lowest  $N$  states according to Pauli's principle. The resulting current is a linear function of rotation frequency and, at large  $\Omega$  can be either positive (eg. with 3 fermions) or negative (eg. with 2 fermions) as can be checked from Fig. 1b. Therefore, the fast mode may disappear even when there are only two spin polarized fermions. This disappearance of the fast mode is due to the Pauli principle, because Pauli principle precludes occupying the same quantum state, including the ground state, by identical fermions. This will be further discussed in the following.

When the fermions are not spin polarized, there can be s-wave interactions. The simplest example to include this interaction (see Eq. 2) is to have one spin up fermion and one spin down fermion in the ring lattice. Fig. 2a shows the current per fermion as a function of rotation frequency at various interactions for the two fermions. Similar to the single atom case, the current approaches constant  $2t$  as the rotation frequency  $\Omega$  increases. However, this asymptotic process depends on the interaction and can be very slow. In general, the current is not necessarily a linear function of rotation frequency. It is important to note that because of the increased dimension of the Hilbert space, there are more level crossings than that for single atom as shown in Fig. 2. Although there maybe accidental degeneracies, generally the rotation frequencies at which level crossings happen do not coincide

for different  $U$ . At large rotation frequency (shown in Fig. 2b), the current is always positive, that is, the current changes continuously from noninteracting case to strongly interacting regime, to be compared with a system with more fermions discussed later in this paper. With even larger  $\Omega$ , the current will approach  $2t$  for all  $U$  in this non-spin-polarized two-atom system.

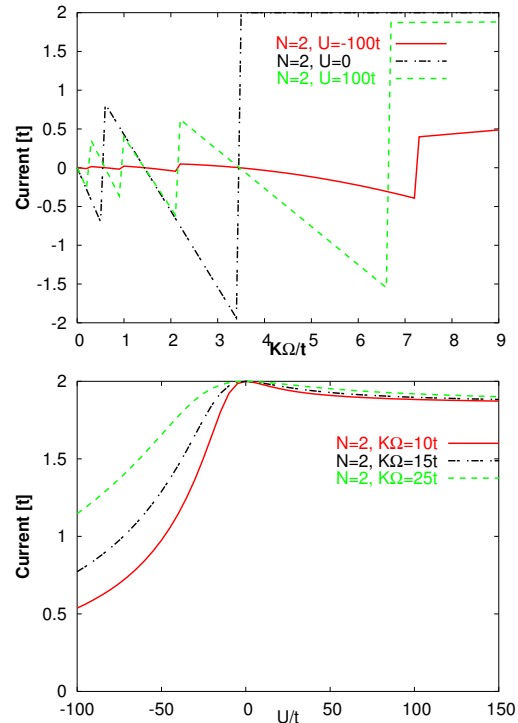


FIG. 2: Numerically calculated current per fermion when one spin up fermion and one spin down fermion are rotated in an eight site-ring lattice. The current per atom always approaches  $2t$  when the rotation frequency is large enough.

It is interesting to study combined effects of spin and interaction by putting four fermions with two up spins and two down spins in the lattice. Figure 3 shows the current as a function of the interaction at large rotation frequencies  $\Omega$ . With both strong attractive and repulsive interactions, positive currents and thus the fast mode persist. However, when the interaction strength is small  $U \sim 0$ , the currents surprisingly become negative and are nearly linear in the rotation frequency. The case without interaction,  $U = 0$ , is easy to understand, because the current is the sum over the currents of the lowest two states for spin up and spin down fermions shown in Fig. 1b. From the discussion on the spin polarized fermions, it is clear that the total current for the four noninteracting fermions is negative (thus not in fast mode) when  $\Omega$  is very large. Furthermore, Fig. 3 also shows that the absence of fast mode for non-spin polarized fermions extends to finite attractive and repulsive interactions. This means that the Pauli principle may suppress the fast mode in many-fermion system even at

nonzero interaction.

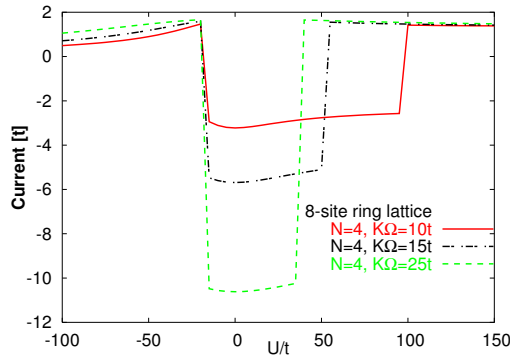


FIG. 3: Fast rotation of four fermions in 8 site-ring lattice. It shows numerically the effects of interaction and spin on the quantum phase transition between fast mode and non-fast mode. The non-fast mode around  $U \sim 0$  is caused by Fermi pressure (see text).

Next, we try to explain the fast mode in such four-fermion system. At large attractive interaction ( $U \ll 0$ ), the fermions form local pairs [22], and thus behave like bosons, so the current is positive which is not surprising given that fast rotating bosons always display a fast mode. Since the fermions get paired roughly independent of the rotation frequency, the transition from the fast mode to the non-fast mode happens at roughly the same  $U$ , independent of the rotating frequency as shown in Fig. 3. While with large repulsive interaction ( $U \gg 0$ ), no such pairing exists and whether the transition happens at  $U$  depends on the rotation frequency. The larger  $\Omega$  is,

the smaller repulsive  $U$  is required to make the transition happen (see Fig. 3). Furthermore, since the fermions are not paired for strong repulsive interaction, the existence of fast modes at large rotation frequency means that the fast mode does not necessarily mean pairing or condensation of fermions.

Although our study is meant to be an in-principle theoretical discussion, a few comments about experiments are in order. The realization of a ring lattice with a tunable boundary phase twist has been proposed recently [23]. Since many bosons can be in the same fast mode, time of flight after turning off the potential of even a single lattice ring should allow detection of the momentum distribution, and thus the fast mode, of the atoms. Detecting the fast mode of fermions may need additional twists, such as using multiple lattice rings, to enhance the signals. Using multiple lattices to enhance the signal has been demonstrated recently [24].

To conclude, we have studied the rotation of bosonic and fermionic atoms in one dimensional lattice rings. We found that minimizing the ground state energy may give the fast mode and the transition to the fast mode in lattice rings is very different from that in continuum rings. Fermionic interaction and the filling factor are shown to have significant effects on the transitions to the fast mode and Fermi pressure may suppress the fast mode. Finally, the fast mode is due to the finite number of lattice sites and is not associated to bosonic statistics, pairing of fermions, or superfluidity.

This work is supported by NSF and Research Corporation. T. Wang acknowledges the helpful discussions with J. Javanainen and U. Shrestha.

- 
- [1] C. A. Regal, C. Ticknor, J. L. Bohn, and D. S. Jin, *Nature* **424**, 47 (2003).
  - [2] M. W. Zwierlein, C. A. Stan, C. H. Schunck, S. M. F. Raupach, A. J. Kerman, and W. Ketterle, *Phys. Rev. Lett.* **92**, 120403 (2004).
  - [3] Q. Chen and K. Levin, *Phys. Rev. Lett.* **95**, 260406 (2005).
  - [4] M. W. J. Romans and H. T. C. Stoof, *Phys. Rev. Lett.* **95**, 260407 (2005).
  - [5] Y. Ohashi and A. Griffin, *Phys. Rev. Lett.* **89**, 130402 (2002).
  - [6] M. W. Zwierlein, J. R. Abo-Shaeer, A. Schirotzek, C. H. Schunck, and W. Ketterle, *Nature* **435**, 1047 (2005).
  - [7] C. Schunck, M. Zwierlein, A. Schirotzek, and W. Ketterle (2006), cond-mat/0607298.
  - [8] M. Greiner, O. Mandel, T. Esslinger, T. W. Hansch, and I. Bloch, *Nature* **415**, 39 (2002).
  - [9] C. Kittel, *Introduction to solid state physics* (John Wiley & Sons, Inc., New York, 1996), ISBN 0-471-11181-3, 7th ed.
  - [10] G. K. Campbell, J. Mun, M. Boyd, E. W. Streed, W. Ketterle, and D. E. Pritchard, *Phys. Rev. Lett.* **96**, 020406 (2006).
  - [11] C. Raman, J. R. Abo-Shaeer, J. M. Vogels, K. Xu, and W. Ketterle, *Phys. Rev. Lett.* **87**, 210402 (2001).
  - [12] A. Aftalion, X. Blanc, and J. Dalibard, *Phys. Rev. A* **71**, 023611 (2005).
  - [13] T.-L. Ho and C. V. Ciobanu, *Phys. Rev. Lett.* **85**, 4648 (2000).
  - [14] N. R. Cooper, S. Komineas, and N. Read, *Phys. Rev. A* **70**, 033604 (2004).
  - [15] H. Pu, L. O. Baksmaty, S. Yi, and N. P. Bigelow, *Phys. Rev. Lett.* **94**, 190401 (2005).
  - [16] C. Wu, H.-d. Chen, J.-p. Hu, and S.-C. Zhang, *Phys. Rev. A* **69**, 043609 (2004).
  - [17] R. Bhat, M. J. Holland, and L. D. Carr, *Phys. Rev. Lett.* **96**, 060405 (2006).
  - [18] R. Bhat, B. M. Peden, B. T. Seaman, M. Krmer, L. D. Carr, and M. J. Holland, *Phys. Rev. A* **74**, 063606 (2006).
  - [19] T. Wang and S. F. Yelin (2006), quant-ph/0610114.
  - [20] E. Merzbacher, *Amer. J. Phys.* **30**, 237 (1962).
  - [21] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, *Phys. Rev. Lett.* **81**, 3108 (1998).
  - [22] L. D. Carr and M. J. Holland, *Phys. Rev. A* **72**, 031604 (2005).
  - [23] L. Amico, A. Osterloh, and F. Cataliotti,

- Phys. Rev. Lett. **95**, 063201 (2005).  
[24] J. Sebby-Strabley, M. Anderlini, P. S. Jessen, and J. V. Porto, Phys. Rev. A **73**, 033605 (2006).